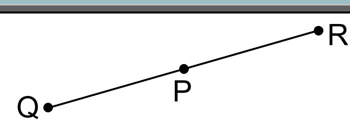
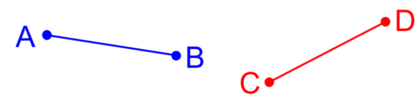
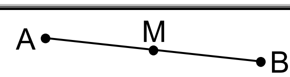
Properties From Algebra

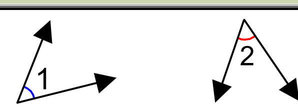
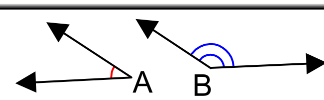
Let a, b, and c represent real numbers. Then:

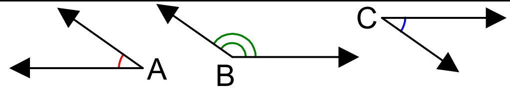
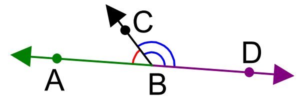
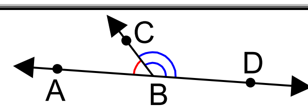
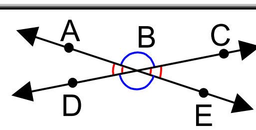
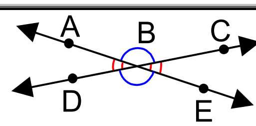
- the Addition Property of Equality says that $a = b$ if and only if $a + c = b + c$.
- the Subtraction Property of Equality says that $a = b$ if and only if $a - c = b - c$.
- the Multiplication Property of Equality says that $a = b$ if and only if $ac = bc$ (where $c \neq 0$).
- the Division Property of Equality says that $a = b$ if and only if $\frac{a}{c} = \frac{b}{c}$ (where $c \neq 0$).
- the Addition Property of Inequality says that $a < b$ if and only if $a + c < b + c$.
- the Subtraction Property of Inequality says that $a < b$ if and only if $a - c < b - c$.
- the Multiplication Property of Inequality says that, if $a < b$ and $c > 0$, then $ac < bc$; if $a < b$ and $c < 0$, then $ac > bc$.
- the Division Property of Inequality says that, if $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$; if $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$.
- the Reflexive Property of Equality says that $a = a$.
- the Symmetric Property of Equality says that $a = b$ if and only if $b = a$.
- the Transitive Property of Equality says that, if $a = b$ and $b = c$, then $a = c$.
- the Substitution Property of Equality says that, if $a = b$, then a may be replaced by b in any equation or inequality.
- the Symmetric Property of Inequality says that $a < b$ if and only if $b > a$.
- the Transitive Property of Inequality says that, if $a < b$ and $b < c$, then $a < c$.
- the Distributive Property says that $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.

Points, Lines, Planes, Line Segments, and Rays

Definition of a midpoint: A point M is called a midpoint of $\overline{AB}$ if and only if it meets both of the following conditions: (1) M is between A and B, and (2) AM = MB	 Given: P is the midpoint of $\overline{QR}$ Then: P is between Q and R and QP = PR AND Given: P is between Q and R and QP = PR Then: P is the midpoint of $\overline{QR}$

Definition of congruent segments: Two line segments are called <u>congruent</u> if and only if the two segments have the same length. The symbol "≅" means "is congruent to."	 Given: $\overline{AB} \cong \overline{CD}$ Then: $AB = CD$ AND Given: $AB = CD$ Then: $\overline{AB} \cong \overline{CD}$ **Note that we do **NOT** say $\overline{AB} = \overline{CD}$. This is because we are referring to pictures (and not numbers) when we write $\overline{AB}$ and $\overline{CD}$, and it does not make sense to say that two pictures are equal to each other.**
Through any two points there is exactly one line.	 Given: Points A and B Then: There is one (and only one) line that contains both A and B.
Midpoint Theorem: If M is the midpoint of $\overline{AB}$, then $\overline{AM} \cong \overline{MB}$.	 Given: M is the midpoint of $\overline{AB}$ Then: $\overline{AM} \cong \overline{MB}$

Angles (You may want to visit our website, <u>www.AlgebraForHomeschool.com</u>, to see a short video that shows you how you can use a protractor to find the measure of an angle.)	
Definition of congruent angles: Two angles are called <u>congruent</u> if and only if their measures are equal.	 Given: $m\angle 1 = m\angle 2$ Then: $\angle 1 \cong \angle 2$ AND Given: $\angle 1 \cong \angle 2$ Then: $m\angle 1 = m\angle 2$ **Note that we do **NOT** say that $\angle 1 = \angle 2$. This is because we are referring to pictures (and not numbers) when we write $\angle 1$ and $\angle 2$, and it does not make sense to say that two pictures are equal to each other.**
Definition of supplementary angles: Two angles are called <u>supplementary</u> if and only if the sum of their measures is 180º.	 Given: $m\angle A + m\angle B = 180^\circ$ Then: $\angle A$ and $\angle B$ are supplementary angles AND Given: $\angle A$ and $\angle B$ are supplementary angles Then: $m\angle A + m\angle B = 180^\circ$

Congruent Supplements Theorem: If two angles are supplementary to the same angle, then they are congruent.	 Given: $\angle A$ and $\angle C$ are both supplementary to $\angle B$ Then: $\angle A \cong \angle C$
Definition of a linear pair: Two angles form a <u>linear pair</u> if and only if they meet both of the following conditions. (1) They are adjacent angles (meaning that the angles share a common side, but the angles do not have any interior points in common), and (2) The sides that the angles do not have in common are opposite rays.	 Given: $\angle ABC$ and $\angle CBD$ form a linear pair Then: $\angle ABC$ and $\angle CBD$ are adjacent angles, and $\overrightarrow{BA}$ and $\overrightarrow{BD}$ are opposite rays AND Given: $\angle ABC$ and $\angle CBD$ are adjacent angles, and $\overrightarrow{BA}$ and $\overrightarrow{BD}$ are opposite rays Then: $\angle ABC$ and $\angle CBD$ form a linear pair
Linear Pair Postulate: If two angles form a linear pair, then they are supplementary.	 Given: $\angle ABC$ and $\angle CBD$ form a linear pair Then: $\angle ABC$ and $\angle CBD$ are supplementary
Definition of vertical angles: Two angles are called <u>vertical angles</u> if and only if they are formed by two pairs of opposite rays.	 Given: $\angle ABC$ and $\angle DBE$ are vertical angles Then: $\overrightarrow{BA}$ and $\overrightarrow{BE}$ are opposite rays; $\overrightarrow{BC}$ and $\overrightarrow{BD}$ are opposite rays AND Given: $\overrightarrow{BA}$ and $\overrightarrow{BE}$ are opposite rays; $\overrightarrow{BC}$ and $\overrightarrow{BD}$ are opposite rays Then: $\angle ABC$ and $\angle DBE$ are vertical angles; $\angle ABD$ and $\angle CBE$ are vertical angles
Vertical Angles Theorem: If two angles are vertical angles, then they are congruent to each other.	 Given: $\angle ABC$ and $\angle DBE$ are vertical angles Then: $\angle ABC \cong \angle DBE$

Example 1: Given that $4x + 3 = -17$, prove that x must equal -5.

Statements	Reasons	Explanations
1. $4x + 3 = -17$	1. Given	We were told that this information is true.
2. $4x = -20$	2. Subtraction Property of Equality	The Subtraction Property of Equality allows us to subtract 3 from both sides of the equation.
3. $x = -5$	3. Division Property of Equality	The Division Property of Equality allows us to divide both sides of the equation by 4.

Example 2: Given that $5y - 2 = 3(y + 4)$, prove that y must equal 7.

(Proof #1)

Statements	Reasons	Explanations
1. $5y - 2 = 3(y + 4)$	1. Given	We were told that this information is true.
2. $5y - 2 = 3y + 12$	2. Distributive Property	The Distributive Property tells us that $3(y + 4) = 3y + 12$.
3. $2y - 2 = 12$	3. Subtraction Property of Equality	The Subtraction Property of Equality allows us to subtract 3y from both sides of the equation.
4. $2y = 14$	4. Addition Property of Equality	The Addition Property of Equality allows us to add 2 to both sides of the equation.
5. $y = 7$	5. Division Property of Equality	The Division Property of Equality allows us to divide both sides of the equation by 2.

(Proof #2)

Statements	Reasons	Explanations
1. $5y - 2 = 3(y + 4)$	1. Given	We were told that this information is true.
2. $5y - 2 = 3y + 12$	2. Distributive Property	The Distributive Property tells us that $3(y + 4) = 3y + 12$.
3. $-2 = -2y + 12$	3. Subtraction Property of Equality	The Subtraction Property of Equality allows us to subtract 5y from both sides of the equation.
4. $-14 = -2y$	4. Subtraction Property of Equality	The Subtraction Property of Equality allows us to subtract 12 from both sides of the equation.
5. $7 = y$	5. Division Property of Equality	The Division Property of Equality allows us to divide both sides of the equation by -2.
6. $y = 7$	6. Symmetric Property of Equality	The Symmetric Property of Equality says that, if we know that $a = b$, then we can say that $b = a$.

Example 3: Given that 16 + 3(4 – 9b) < 3b – 2, prove that b > 1.

(Proof #1)

Statements	Reasons	Explanations
1. 16 + 3(4 – 9b) < 3b – 2	1. Given	We were told that this information is true.
2. 16 + 12 – 27b < 3b – 2	2. Distributive Property	The Distributive Property tells us that 3(4 – 9b) = 12 – 27b.
3. 28 – 27b < 3b – 2	3. Substitution Property of Equality	The Substitution Property of Equality tells us that we can substitute 28 for 16 + 12.
4. 28 – 30b < –2	4. Subtraction Property of Inequality	The Subtraction Property of Inequality allows us to subtract 3b from both sides of the inequality.
5. –30b < –30	5. Subtraction Property of Inequality	The Subtraction Property of Inequality allows us to subtract 28 from both sides of the inequality.
6. b > 1	6. Division Property of Inequality	The Division Property of Inequality allows us to divide both sides of the inequality by –30 as long as we flip the inequality symbol.

(Proof #2)

Statements	Reasons	Explanations
1. 16 + 3(4 – 9b) < 3b – 2	1. Given	We were told that this information is true.
2. 16 + 12 – 27b < 3b – 2	2. Distributive Property	The Distributive Property tells us that 3(4 – 9b) = 12 – 27b.
3. 28 – 27b < 3b – 2	3. Substitution Property of Equality	The Substitution Property of Equality tells us that we can substitute 28 for 16 + 12.
4. 28 < 30b – 2	4. Addition Property of Inequality	The Addition Property of Inequality allows us to add 27b to both sides of the inequality.
5. 30 < 30b	5. Addition Property of Inequality	The Addition Property of Inequality allows us to add 2 to both sides of the inequality.
6. 1 < b	6. Division Property of Inequality	The Division Property of Inequality allows us to divide both sides of the inequality by 30.
7. b > 1	7. Symmetric Property of Inequality	The Symmetric Property of Inequality says that a < b if and only if b > a.

Example 3: Given that $16 + 3(4 – 9b) < 3b – 2$, prove that $b > 1$.

(Proof #3)

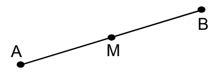
Statements	Reasons	Explanations
1. $16 + 3(4 – 9b) < 3b – 2$	1. Given	We were told that this information is true.
2. $3(4 – 9b) < 3b – 18$	2. Subtraction Property of Inequality	The Subtraction Property of Inequality allows us to subtract 16 from both sides of the inequality.
3. $4 – 9b < b – 6$	3. Division Property of Inequality	The Division Property of Inequality allows us to divide both sides of the inequality by 3.
4. $–9b < b – 10$	4. Subtraction Property of Inequality	The Subtraction Property of Inequality allows us to subtract 4 from both sides of the inequality.
5. $–10b < –10$	5. Subtraction Property of Inequality	The Subtraction Property of Inequality allows us to subtract b from both sides of the inequality.
6. $b > 1$	6. Division Property of Inequality	The Division Property of Inequality allows us to divide both sides of the inequality by –10 as long as we flip the inequality symbol.

(Proof #4)

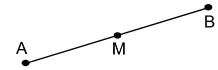
Statements	Reasons	Explanations
1. $16 + 3(4 – 9b) < 3b – 2$	1. Given	We were told that this information is true.
2. $16 + 12 – 27b < 3b – 2$	2. Distributive Property	The Distributive Property tells us that $3(4 – 9b) = 12 – 27b$.
3. $18 + 12 – 27b < 3b$	3. Addition Property of Inequality	The Addition Property of Inequality tells us that we can add 2 to both sides of the inequality.
4. $18 + 12 < 30b$	4. Addition Property of Inequality	The Addition Property of Inequality allows us to add 27b to both sides of the inequality.
5. $30 < 30b$	5. Substitution Property of Equality	The Substitution Property of Equality allows us to substitute 30 for 18 + 12.
6. $1 < b$	6. Division Property of Inequality	The Division Property of Inequality allows us to divide both sides of the inequality by 30.
7. $b > 1$	7. Symmetric Property of Inequality	The Symmetric Property of Inequality says that $a < b$ if and only if $b > a$.

Example 4: Prove the Midpoint Theorem. This theorem says that, if M is the midpoint of $\overline{AB}$, then $\overline{AM} \cong \overline{MB}$.

We must begin by drawing a generic picture. When we want to prove a theorem, this picture should show the "if" part of the theorem.



Example 4: Prove the Midpoint Theorem. This theorem says that, if M is the midpoint of $\overline{AB}$, then $\overline{AM} \cong \overline{MB}$.

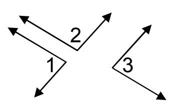


When we want to prove a theorem, the "if" part is the information we are given, and the "then" part is the part we want to prove. Therefore, for this proof, we can say that we are given that M is the midpoint of $\overline{AB}$, and we want to prove that $\overline{AM} \cong \overline{MB}$.

Statements	Reasons	Explanations
1. M is the midpoint of $\overline{AB}$	1. Given	We were told that this information is true.
2. AM = MB	2. Definition of midpoint	The definition of midpoint says that, if you start out knowing that a point M is the midpoint of $\overline{AB}$, then AM = MB.
3. $\overline{AM} \cong \overline{MB}$	3. Definition of congruent segments	The definition of congruent segments says that, if you start out knowing that you have segments that have equal lengths, then they are congruent.

Example 5: Prove the Congruent Supplements Theorem. This theorem says that, if two angles are supplementary to the same angle, then they are congruent.

We must begin by drawing a generic picture that describes the "if" part of this statement.



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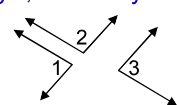
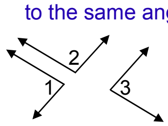


Now, we can say that we are given that $\angle 2$ is supplementary to both $\angle 1$ and $\angle 3$, and we want to prove that $\angle 1 \cong \angle 3$.

Example 5: Prove the Congruent Supplements Theorem. This theorem says that, if two angles are supplementary to the same angle, then they are congruent.



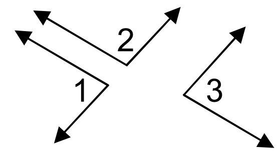
Given: $\angle 2$ is supplementary to both $\angle 1$ and $\angle 3$

Prove: $\angle 1 \cong \angle 3$

Before we do a formal proof, it is often useful to convince ourselves that the statement is actually true. So let's pick some numbers for the measure of $\angle 1$ and see what happens.

However, it is important to realize that this does not prove that $\angle 1$ is ALWAYS congruent to $\angle 3$ in this situation. It is possible that we just happened to pick some very special numbers.

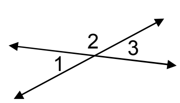
Example 5: Prove the Congruent Supplements Theorem. This theorem says that, if two angles are supplementary to the same angle, then they are congruent.



Statements	Reasons	Explanations
1. ∠2 is supplementary to both ∠1 and ∠3	1. Given	We were told that this information is true.
2. m∠1 + m∠2 = 180º and m∠2 + m∠3 = 180º	2. Definition of supplementary angles	The definition of supplementary angles tells us that, if we start out knowing that two angles are supplementary, then the sum of their measures must equal 180º.
3. m∠1 + m∠2 = m∠2 + m∠3	3. Substitution Property of Equality	Since we said in Statement (2) that m∠2 + m∠3 = 180º, we can use the Substitution Property of Equality to substitute m∠2 + m∠3 for 180º in the equation m∠1 + m∠2 = 180º.
4. m∠1 = m∠3	4. Subtraction Property of Equality	The Subtraction Property of Equality allows us to subtract m∠2 from both sides of the equation in Statement (3).
5. ∠1 ≅ ∠3	5. Definition of congruent angles	The definition of congruent angles tells us that, if you know that the measures of two angles are equal, then they must be congruent.

Example 6: Prove the Vertical Angles Theorem. This theorem says that, if two angles are vertical angles, then they are congruent.

We must begin by drawing a generic picture that describes the "if" part of this statement.



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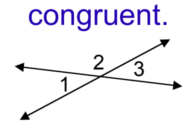
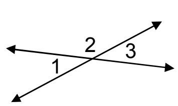


Now, we can say that we are given that ∠1 and ∠3 are vertical angles, and we want to prove that ∠1 ≅ ∠3.

Example 6: Prove the Vertical Angles Theorem. This theorem says that, if two angles are vertical angles, then they are congruent.

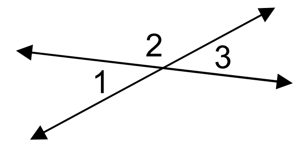


Given: ∠1 and ∠3 are vertical angles

Prove: ∠1 ≅ ∠3

Before we do a formal proof, it is often useful to do some experimentation and hopefully come up with a plan.

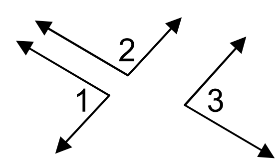
Example 6: Prove the Vertical Angles Theorem. This theorem says that, if two angles are vertical angles, then they are congruent.



Statements	Reasons	Explanations
1. ∠1 and ∠3 are vertical angles	1. Given	We were told that this information is true.
2. ∠1 and ∠2 form a linear pair, and ∠2 and ∠3 form a linear pair	2. Definition of linear pair	We can look at the picture and use the definition of a linear pair to see that ∠1 and ∠2 form a linear pair and that ∠2 and ∠3 form a linear pair.
3. ∠1 and ∠2 are supplementary, and ∠2 and ∠3 are supplementary	3. Linear Pair Postulate	The Linear Pair Postulate says that, if two angles form a linear pair, then they are supplementary. Since we said in Statement (2) that ∠1 and ∠2 form a linear pair and that ∠2 and ∠3 form a linear pair, we can use this postulate to say that ∠1 and ∠2 are supplementary and that ∠2 and ∠3 are supplementary.
4. ∠1 ≅ ∠3	4. Congruent Supplements Theorem	Since we said in Statement (3) that ∠2 is supplementary to both ∠1 and ∠3, we can use the Congruent Supplements Theorem to say that ∠1 ≅ ∠3.

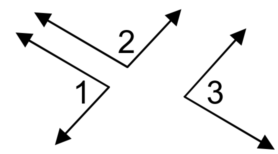
The proofs we have looked at so far have been in what we call a two-column format. Proofs can also be written in paragraph form or in the form of a flow chart.

Example 7: Prove the Congruent Supplements Theorem using a paragraph proof.



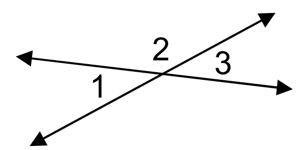
We are given that ∠1 and ∠3 are both supplementary to ∠2. The definition of supplementary angles tells us that this means that m∠1 + m∠2 = 180º and that m∠2 + m∠3 = 180º. Now, we can use the Substitution Property of Equality to say that m∠1 + m∠2 = m∠2 + m∠3. This means that we can use the Subtraction Property of Equality to say that m∠1 = m∠3. We can now use the definition of congruent angles to conclude that ∠1 ≅ ∠3.

Example 8: Prove the Congruent Supplements Theorem using a flow proof.



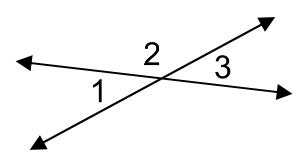
$\angle 1$ is supplementary to $\angle 2$	Given
$m\angle 1 + m\angle 2 = 180°$	Definition of supplementary angles
$\angle 2$ is supplementary to $\angle 3$	Given
$m\angle 2 + m\angle 3 = 180°$	Definition of supplementary angles
$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$	Substitution Property of Equality
$m\angle 1 = m\angle 3$	Subtraction Property of Equality
$\angle 1 \cong \angle 3$	Definition of congruent angles

Example 9: Prove the Vertical Angles Theorem using a paragraph proof.



We are given that $\angle 1$ and $\angle 3$ are vertical angles. The definition of linear pair tells us that $\angle 1$ and $\angle 2$ form a linear pair and that $\angle 2$ and $\angle 3$ form a linear pair. This means we can use the Linear Pair Postulate to say that $\angle 1$ and $\angle 2$ are supplementary and that $\angle 2$ and $\angle 3$ are supplementary. Then, by the Congruent Supplements Theorem, $\angle 1$ must be congruent to $\angle 3$.

Example 10: Prove the Vertical Angles Theorem using a flow proof.

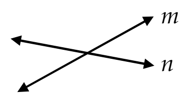


$\angle 1$ and $\angle 3$ are vertical angles	Given
$\angle 1$ and $\angle 2$ form a linear pair	Definition of linear pair
$\angle 1$ and $\angle 2$ are supplementary	Linear Pair Postulate
$\angle 2$ and $\angle 3$ form a linear pair	Definition of linear pair
$\angle 2$ and $\angle 3$ are supplementary	Linear Pair Postulate
$\angle 1 \cong \angle 3$	Congruent Supplements Theorem

We can also prove statements using a type of proof called <u>proof by contradiction</u>. These proofs are also called <u>indirect proofs</u>.
To prove a statement by contradiction, start by assuming the given information and the negation (or opposite) of the statement you're trying to prove.

Example 11: Use an indirect proof to show that, if two distinct lines intersect, then they intersect in exactly one point.

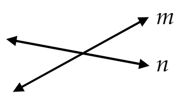
Of course, we must begin by drawing a picture and deciding what we are given and what we want to prove.



Given: Lines m and n are two distinct lines that intersect

Prove: The intersection of lines m and n is exactly one point

Example 11: Use an indirect proof to show that, if two distinct lines intersect, then they intersect in exactly one point.

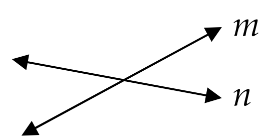


Given: Lines m and n are two distinct lines that intersect

Prove: The intersection of lines m and n is exactly one point

We will begin by noting that the definition of intersection tells us that the lines will intersect in *at least* one point, and so all we need to show is that they cannot intersect in more than one point.

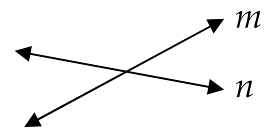
Example 11: Use an indirect proof to show that, if two distinct lines intersect, then they intersect in exactly one point.



(Two-Column Proof)

Statements	Reasons	Explanations
1. Lines m and n are two distinct lines that intersect.	1. Given	We were told that this information is true.
2. Assume that lines m and n intersect in at least two points. Call these points A and B.	2. Assumption	When we want to prove a statement using an indirect proof, we assume the given information and the negation of the statement we want to prove.
3. Lines m and n both contain the points A and B.	3. Definition of the intersection of two figures	We said in Statement (2) that lines m and n intersect in both points A and B. Then, by the definition of the intersection of two figures, lines m and n must contain both point A and point B.
4. Lines m and n intersect in exactly one point.	4. Contradiction	We said in Statement (3) that lines m and n both contain points A and B. However, this cannot happen because the postulate *Through any two points there is exactly one line* tells us that there is only one line that passes through both A and B.

Example 11: Use an indirect proof to show that, if two distinct lines intersect, then they intersect in exactly one point.



(Paragraph Proof)

We are given that lines m and n are distinct lines that intersect in at least one point. Assume that they intersect in at least two points. We will call these points A and B. Then, by the definition of the intersection of two figures, lines m and n must contain both point A and point B. This presents a contradiction because the postulate *Through any two points there is exactly one line* tells us that there is only one line that passes through both point A and point B. Therefore, lines m and n must intersect in exactly one point.

(Flow Proof)

Lines m and n are two distinct lines that intersect	→	Assume that lines m and n intersect in at least two points. Call these points A and B.	→	Lines m and n both contain points A and B.	→	Lines m and n intersect in exactly one point
Given		Assumption		Definition of the intersection of two figures		Contradiction